

Random elastic networks : strong disorder renormalization approach

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For arbitrary networks of random masses connected by random springs, we define a general strong disorder real-space renormalization (RG) approach that generalizes the procedures introduced previously by Hastings [Phys. Rev. Lett. 90, 148702 (2003)] and by Amir, Oreg and Imry [Phys. Rev. Lett. 105, 070601 (2010)] respectively. The principle is to eliminate iteratively the elementary oscillating mode of highest frequency associated with either a mass or a spring constant. To explain the accuracy of the strong disorder RG rules, we compare with the Aoki RG rules that are exact at fixed frequency.

I. INTRODUCTION

In the field of disordered systems, the problem of random masses connected by random springs is very old since it has been introduced by Dyson [1] even before the classical Anderson localization paper [2]. After studies concerning the one-dimensional case (see the review [3] and references therein), an analysis of disordered elastic media via a non-linear sigma-model [4] has predicted results similar to the scaling theory of Anderson localization [5] : all finite-frequency phonons are localized in dimension $d \leq 2$, whereas there exists a finite critical frequency transition between delocalized and localized modes in dimension $d > 2$. More recent discussions on the similarities and differences with Anderson localization of electrons can be found in [6–8] and references therein.

Besides the case of regular lattices, one may also consider the case of more general networks with various physical motivations. Two types of strong disorder real-space renormalization (RG) procedures have been previously introduced: Hastings [9] has proposed to eliminate iteratively masses, whereas Amir, Oreg and Imry [10] have proposed to eliminate iteratively springs. In these two studies, the results have been found to be extremely accurate at low energies. In the present paper, we unify these two particular procedures into a single framework, where both masses or springs can be iteratively eliminated in a consistent way. In addition, we explain the accuracy of strong disorder RG procedures at low frequency in these models, by a comparison with the Aoki RG rules that are exact at fixed frequency (this type of renormalization has been first introduced by Aoki in the context of Anderson localization in [11–13]).

The paper is organized as follows. Section II presents the models and notations. In section III, we describe the strong disorder RG rules consisting in the iterative elimination of the site or the link associated with the highest frequency, and explain the relations with the previous approaches [9, 10]. In section IV, we compare with the Aoki RG rules that are exact at fixed frequency. Our conclusions are summarized in section V.

II. DISORDERED NETWORK OF MASSES CONNECTED BY SPRINGS

We consider an arbitrary network where sites are indexed by i . With each site is associated a random mass m_i , and with each link $\langle i, j \rangle$ is associated a random spring constant $K_{i,j} = K_{j,i}$.

The scalar phonon model is defined by the following harmonic Hamiltonian for the scalar displacements $u_i(t)$

$$H = \sum_i \frac{m_i}{2} \dot{u}_i^2 + \sum_{\langle i,j \rangle} \frac{K_{i,j}}{2} (u_i - u_j)^2 \quad (1)$$

The scalar assumption is very standard to simplify the analysis [7] and means physically that vibrations along different directions are decoupled. Equivalently, the model can be defined by the equations of motion

$$m_i \ddot{u}_i = \sum_j K_{i,j} (u_j - u_i) \quad (2)$$

As stressed in [9, 10], this scalar phonon formulation can be used to study various other relevant physical models :

(i) the case $m_i = 1$ with random $K_{i,j}$ can be used to study the Laplacian on the network with disordered couplings between nodes [9, 10]. For instance in [10], the sites were drawn at random in a d -dimensional cube and the $K_{i,j}$ was exponentially decaying in the Euclidean distance $r_{i,j}$. In [9], the motivation was coming from the field of complex networks (see the recent reviews [14]).

(ii) the case $m_i = 1/n_i$ where n_i represents the number of nodes connected to i and $K_{i,j} = 1$ when (i,j) are neighbors on the network (with $K_{i,j} = 0$ otherwise) corresponds to the usual random walk on the network. Here the 'disorder' comes only from the network heterogeneous structure.

III. STRONG DISORDER RENORMALIZATION (R.G.) APPROACH

Strong disorder renormalization (see [15] for a review) is a very specific type of RG that has been first developed in the field of quantum spins : the RG rules of Ma and Dasgupta [16] have been put on a firm ground by D.S. Fisher who introduced the crucial idea of "infinite disorder" fixed point where the method becomes asymptotically exact, and who computed explicitly exact critical exponents and scaling functions for one-dimensional disordered quantum spin chains [17]. This method has thus generated a lot of activity for various disordered quantum models : see the review [15] for works before 2004, as well as more recent developments concerning entanglement [18, 19], superfluid-insulator transition of disordered bosons [20], Bose-Einstein condensation [21], dissipation effects [22, 23], Hubbard model [24], anyonic chains [25], fractal lattices [26], studies on the link with extremal statistics [27], and implementation of new very efficient numerical procedures [28]. Strong disorder renormalization has been also successfully applied to various classical disordered dynamical models, such as random walks in random media either in one dimension [29, 30], on strips [31], in two dimensions [32], or in the presence of absorbers [33], reaction-diffusion [34], coarsening dynamics of classical spin chains [35], trap models [36], random vibrational networks [9, 10], contact processes [38, 39], zero range processes [40] and exclusion processes [41], non-equilibrium dynamics of polymers or interfaces in random media [42], statistics of valleys in configuration space of disordered systems [43], oscillator synchronisation [44], and extreme value statistics of various stochastic processes [45].

In this section, we describe a strong disorder renormalization procedure for the network of masses connected by springs of Eq. 1 and 2 that generalizes the previous procedures proposed in [9, 10]. The principle of this RG procedure is to eliminate iteratively the highest frequency present in the system. The elementary oscillating modes are of two types as we now discuss.

A. Oscillating modes associated with a single spring

Let us first consider the problem of a single spring between two masses m_1 and m_2 , when all other spring constants are supposed to be negligible. The equations of motion

$$m_1 \ddot{u}_1 + K_{1,2} (u_1 - u_2) = 0 \quad (3)$$

$$m_2 \ddot{u}_2 + K_{1,2} (u_2 - u_1) = 0 \quad (4)$$

can be analyzed by introducing the center-of-mass displacement u_G and the relative displacement u_{rel}

$$u_G \equiv \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \quad (5)$$

$$u_{rel} \equiv u_2 - u_1 \quad (6)$$

as well as their corresponding masses

$$m_G \equiv m_1 + m_2 \quad (7)$$

$$m_{rel} \equiv \frac{m_1 m_2}{m_1 + m_2} \quad (8)$$

The equations of motion of Eq. 4 are then decoupled

$$m_G \ddot{u}_G = 0 \quad (9)$$

$$m_{rel} \ddot{u}_{rel} + K_{1,2} u_{rel} = 0 \quad (10)$$

One obtains the zero-mode corresponding to the global motion of the center of mass, and the oscillating model of the relative coordinate at the frequency Ω_{rel} given by

$$\Omega_{1,2}^2 = \frac{K_{1,2}}{m_{rel}} = K_{1,2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \quad (11)$$

If this frequency is high, this mode will not be excited by the low-energy modes of the whole system : one may then eliminate the corresponding relative mode, i.e. we set

$$u_{rel} = u_2 - u_1 = 0 \quad (12)$$

The masses m_1 and m_2 are then merged into their center of mass G of mass $m_G = m_1 + m_2$ and of displacement $u_G = u_1 = u_2$. The equation of motion for the center of mass now takes into account the forces of other points $i \neq (1, 2)$ that were connected either to (1) or to (2) or to both

$$m_G \ddot{u}_G = \sum_{i \neq (1,2)} (K_{i,1} + K_{i,2}) (u_i - u_G) \quad (13)$$

that had been neglected in Eq. 10 in the single spring analysis. The equation of motion for $i \neq (1, 2)$ becomes

$$m_i \ddot{u}_i = \sum_{n \neq (1,2)} K_{i,n} (u_n - u_i) + (K_{i,1} + K_{i,2}) (u_G - u_i) \quad (14)$$

In summary, the elimination of a spring $K_{1,2}$ associated with a high frequency $\Omega_{1,2}$ (see Eq. 11) consists in the merging of the two sites 1 and 2 into a single site representing their center of mass G of mass $m_G = m_1 + m_2$. The approximation of Eq. 12 yields that the springs constant connected to G are given by to the following renormalization rule

$$K_{i,G}^{new} = (K_{i,1} + K_{i,2}) \quad (15)$$

whereas the masses m_i are unchanged.

B. Oscillating mode associated with a single mass

The problem of a single mass m_0 in the environment of the other masses m_j that are supposed to be fixed ($u_i(t) = u_i$ independent of time) is of course very simple : from the equation of motion

$$m_0 \ddot{u}_0 + \left[\sum_i K_{0,i} \right] u_0(t) = \sum_i K_{0,i} u_i \quad (16)$$

one obtains that $u_0(t)$ will oscillate around the equilibrium value

$$u_0^{eq} = \frac{\sum_i K_{0,i} u_i}{\sum_n K_{0,n}} \quad (17)$$

with the frequency Ω_0 given by

$$\Omega_0^2 = \frac{1}{m_0} \sum_i K_{0,i} \quad (18)$$

If this frequency is high, this mode will not be excited in the low-energy modes of the whole system : one may then eliminate it. This amounts to say that $u_0(t)$ will adiabatically follow the slow motion of its neighbors

$$u_0^{adiab}(t) \simeq \frac{\sum_i K_{0,i} u_i(t)}{\sum_n K_{0,n}} \quad (19)$$

The elimination of the mass m_0 leads to the following renormalized of the equation of motion for the other masses $i \neq 0$

$$\begin{aligned} m_i \ddot{u}_i &= \sum_{j \neq 0} K_{i,j} (u_j - u_i) + K_{i,0} \left(\frac{\sum_j K_{0,j} u_j(t)}{\sum_n K_{0,n}} - u_i \right) \\ &= \sum_{j \neq 0} \left[K_{i,j} + \frac{K_{i,0} K_{0,j}}{\sum_n K_{0,n}} \right] (u_j - u_i) \end{aligned} \quad (20)$$

In summary, the elimination of a mass m_0 associated with a high frequency Ω_0 (see Eq. 18) leads via the adiabatic approximation of Eq. 19 to the following renormalizations for the spring constants between two neighbors (i, j) of the mass m_0

$$K_{i,j}^{new} = K_{i,j} + \frac{K_{i,0}K_{0,j}}{\sum_n K_{0,n}} \quad (21)$$

whereas the masses m_i are unchanged.

C. Statement of the strong disorder renormalization procedure

The above analysis suggest the following strong disorder renormalization procedure :

(i) at a given RG step, there are a certain number of masses m_i and a certain number of spring constants $K_{i,j} = K_{j,i}$ (with $i \neq j$).

With each spring $K_{i,j}$, one associates the frequency $\Omega_{i,j}$ given by

$$\Omega_{i,j}^2 \equiv K_{i,j} \left(\frac{1}{m_i} + \frac{1}{m_j} \right) \quad (22)$$

With each mass m_i , one associates the frequency Ω_i given by

$$\Omega_i^2 \equiv \frac{\sum_j K_{i,j}}{m_i} \quad (23)$$

The renormalization scale Ω is defined as the highest frequency Ω remaining in the system

$$\Omega \equiv \max [\{\Omega_i, \Omega_{i,j}\}] \quad (24)$$

among all frequencies associated with masses or spring constants.

(ii) Decimation of the mode associated with the highest frequency Ω :

(ii-a) If the highest frequency $\Omega = \Omega_{i_0,j_0}$ is associated with the spring constant K_{i_0,j_0} : the two masses m_{i_0} and m_{j_0} are replaced by their center of mass $G(i_0, j_0)$ of mass

$$m_{G(i_0,j_0)} = m_{i_0} + m_{j_0} \quad (25)$$

and the spring constants linked to i_0 or to j_0 or to both are replaced by spring constants linked to their center of mass

$$K_{j,G(i_0,j_0)} = K_{j,i_0} + K_{j,j_0} \quad (26)$$

(ii-b) If the highest frequency $\Omega = \Omega_{i_0}$ is associated with the mass i_0 : the mass m_{i_0} is eliminated, and the spring constants between two neighbors (i, j) of (i_0) are renormalized according to

$$K_{i,j}^{new} = K_{i,j} + \frac{K_{i,i_0}K_{i_0,j}}{\sum_n K_{i_0,n}} \quad (27)$$

(iii) Return to point (i) to update the frequencies associated with the surviving renormalized mass and to the surviving renormalized springs.

D. Relation with the RG rules proposed by Amir, Oreg and Imry [10]

In [10], Amir, Oreg and Imry have proposed to choose at each step the largest spring constant $K_{i,j}$ and then to apply RG rules that coincide with Eqs 25 and 26. They insist that one should choose the largest spring constant independently of the masses, and not the highest corresponding frequency that we introduced in Eq. 22. We do not agree on this point, since we believe that the frequency scale is the only well founded physical variable to define a consistent strong disorder procedure for phonons. We believe that for the specific application studied in [10], the distribution of spring constants is so broad whereas all masses are initially equal, that the two choices are probably equivalent. However for other applications, we believe that the frequency choice is the relevant one to compare the possible excitations of various sub-systems. At the beginning of the RG procedure, the considered sub-systems are made of a single mass or of a pair of masses connected by a spring, but as the RG proceeds, the considered sub-systems contain more and more initial masses. The frequency criterion allows to construct in a consistent way the appropriate renormalized modes that will respond to an exterior low-energy excitation.

E. Relation with the RG rules proposed by Hastings [9]

In [9], Hastings first performs the similarity transformation [1]

$$u_i = \frac{v_i}{\sqrt{m_i}} \quad (28)$$

to obtain a symmetric operator from the equation of motion of Eq. 2

$$\ddot{v}_i = - \sum_j L_{i,j} v_j \quad (29)$$

with

$$\begin{aligned} L_{i,i} &\equiv \frac{1}{m_i} \sum_j K_{i,j} \\ L_{i \neq j} &\equiv - \frac{K_{i,j}}{\sqrt{m_i m_j}} \end{aligned} \quad (30)$$

The first RG procedure proposed in [9] consists in choosing the highest $L_{i,i}$ at each step : since $L_{i,i} = \Omega_i^2$ of Eq. 23, this is equivalent to choose the highest frequency associated with masses only (and the frequencies $\Omega_{i,j}^2$ of Eq. 22 associated with links are not considered for this choice). The RG rule corresponding to the elimination of L_{i_0, i_0} reads [9] (after correction of the typo concerning the sign of the second term in [9])

$$L_{i,j}^{new} = L_{i,j} - \frac{L_{i,i_0} L_{i_0,j}}{L_{i_0,i_0}} \quad (31)$$

which is equivalent to the RG rule of Eq. 27 for the spring constants and to keeping the masses unchanged for the remaining sites.

The second RG procedure proposed in [9] still consists in choosing the highest $L_{i,i}$ at each step, but to treat separately its neighbor j with the highest $|L_{i,j}|$: the RG then consists in the diagonalization of this two-by-two submatrix and to project out the eigenvector of highest eigenvalue. In [9], the advantages of this procedure with respect to the previous rule of Eq. 31 were to allow the increase of the connectivity and the variations of the masses. However these properties are also present in the simpler rules of Eqs 25 and 26 concerning the decimation of a link. We thus believe that the RG rules described in section IIIC that consider both the decimations of sites and links is physically clearer and is able to renormalize at the same time spring constants, masses and connectivity of the network in a consistent way.

F. Validity of the strong disorder renormalization procedure

This renormalization procedure will be consistent if the renormalization scale Ω defined in point (i) decreases at each step, i.e. if the new generated frequencies are smaller than the decimated frequency Ω . Moreover from what happens in other fields [15], one expects that the procedure will become asymptotically exact at small Ω , if the renormalized distribution of the frequencies become broader and broader as $\Omega \rightarrow 0$. For an arbitrary network with an arbitrary disorder of masses and spring constants, it is not easy to know a priori if the procedure will remain consistent, and if the procedure will become asymptotically exact at low frequency. However, this can be checked numerically for each case of interest. It turns out that in the various cases studied numerically in [9, 10] the results have been found to be extremely accurate at low energies. In the following section, we explain that this accuracy could come from the coincidence with the zero-frequency limit of Aoki exact RG rules.

IV. COMPARISON WITH AOKI EXACT RENORMALIZATION PROCEDURE AT FIXED FREQUENCY

For Anderson localization models, Aoki [11–13] has introduced an exact real-space renormalization procedure at fixed energy which preserves the Green functions of the remaining sites. This procedure has been further studied for one-particle models in [46–48]. It has been also extended in configuration space for two-particle models [49] and for manybody localization models [50]. This method can be extended to any eigenvalue equation, so it can be applied to the scalar phonon problem at fixed frequency as we now explain.

A. Equations of motion at fixed frequency

Since the equations of motion of Eq. 2 are linear, one may also analyse the dynamics in terms of oscillating modes \hat{u}_ω of a given fixed frequency ω :

$$-m_i \omega^2 \hat{u}_\omega(i) = \sum_j K_{i,j} (\hat{u}_\omega(j) - \hat{u}_\omega(i)) \quad (32)$$

B. Aoki renormalization procedure

Eq. 32 for an arbitrary site $i = i_0$ can be used to eliminate $\hat{u}_\omega(i_0)$ with

$$\hat{u}_\omega(i_0) = \frac{\sum_j K_{i_0,j} \hat{u}_\omega(j)}{\sum_n K_{i_0,n} - m_{i_0} \omega^2} \quad (33)$$

Then the equations of the remaining sites $i \neq i_0$ remain of the same form as Eq. 32

$$-m_i^{new} \omega^2 \hat{u}_\omega(i) = \sum_{j \neq i_0} K_{i,j}^{new}(\omega) (\hat{u}_\omega(j) - \hat{u}_\omega(i)) \quad (34)$$

with the renormalized frequency-dependent springs constants

$$K_{i,j}^{new}(\omega) = K_{i,j}(\omega) + \frac{K_{i,i_0}(\omega) K_{i_0,j}(\omega)}{\sum_n K_{i_0,n}(\omega) - \omega^2 m_{i_0}(\omega)} \quad (35)$$

and the renormalized frequency-dependent masses

$$m_i^{new}(\omega) = m_i(\omega) + \frac{K_{i,i_0}(\omega) m_{i_0}(\omega)}{\sum_n K_{i_0,n}(\omega) - \omega^2 m_{i_0}(\omega)} \quad (36)$$

Let us stress the two essential differences with the strong disorder RG rules discussed in the previous section :

(i) Aoki RG procedure is exact for any order in the choice of the decimated sites i_0 , whereas in strong disorder RG procedure it is essential to choose at each step the mode corresponding to the highest elementary frequency.

(ii) The price to pay is that Aoki RG concerns a fixed frequency ω , and that the renormalized parameters contain this frequency. So for each frequency, one should restart at the very beginning the RG procedure to compute the new renormalized couplings. On the contrary, the renormalized masses and springs of the strong disorder RG procedure do not contain the frequency.

So these two RG procedures are completely different in spirit and in practice. But it turns out that they become similar if one consider the Aoki RG rules in the limit of zero-frequency.

C. Limit of zero-frequency of Aoki RG rules

In the limit of zero frequency $\omega \rightarrow 0$, the elimination of i_0 corresponds to the RG rule (Eq. 35)

$$K_{i,j}^{new}(0) = K_{i,j}(0) + \frac{K_{i,i_0}(0) K_{i_0,j}(0)}{\sum_n K_{i_0,n}(0)} \quad (37)$$

which coincides with the strong disorder RG rule of Eq. 27. Note that the other RG rule of Eq. 36 concerning the masses disappear in the limit $\omega \rightarrow 0$ since all masses appear always together with ω^2 in the equation of motion (Eq. 32).

This coincidence between the strong disorder RG rules and the zero-frequency Aoki exact RG rules may explain the numerical accuracy of the strong disorder RG rules at low frequency found in [9, 10]. Moreover, since Aoki procedure is exact for any order of the decimated sites, one expects that 'bad decimations' during the strong disorder RG ('bad decimations' meaning decimations when the eliminated frequency is not well separated from the others) will actually be corrected in the later stages of the renormalization when the neighbors will themselves be decimated. This phenomenon has already been found in other applications of strong disorder RG, in particular for quantum spin chains (see Appendix E of [17]) and for the construction of valleys in configuration space for disordered systems (see section 3.6 of [43]).

V. CONCLUSIONS AND PERSPECTIVES

In summary, we have proposed a strong disorder renormalization procedure for the random phonon problem on arbitrary networks, that generalizes previous approaches where only site decimations [9] or only link decimations [10] had been considered. We believe that the iterative elimination of the mode of highest frequency associated with either a mass or a spring constant is the appropriate formulation to study cases where both processes can a priori occur during the RG process (in analogy with quantum spin chains in random transverse fields where one decimates either sites or bonds with an energy gap criterion [15, 17]). To explain the accuracy of the RG rules found in the various examples studied numerically in [9, 10], we have compared with the Aoki RG rules that are exact at fixed frequency.

More generally, strong disorder RG rules are naturally defined on arbitrary networks, since even if the starting point is a regular lattice of dimension $d > 1$, the RG rules changes the connectivity and thus destroys the initial regular structure (the one-dimensional lattice is usually the only case where the renormalized structure remains one-dimensional). So they can be applied to disordered systems defined on complex networks. However it turns out that complex networks constitute by themselves a disordered structure, so that even non-disordered models defined on non-regular networks can present Griffiths phases [51] or localization effects [52]. It would be thus interesting in the future to study whether strong disorder RG rules could be appropriate to study non-disordered systems on complex networks that are sufficiently inhomogeneous. It turns out that among the various RG procedures that have been recently introduced for complex networks [53–56], a recent proposal [57] consists in a purely sequential algorithm where at each step, a node is selected at random to perform an elementary RG transformation : however since the nodes are not all equivalent, a natural question is whether the selection of a node at random is really the good choice, or whether one should choose the node or the link according to a maximal criterion, as in strong disorder RG procedures.

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